On a new approach to optical solitons in dielectric fibers

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Abstract

In this letter we describe a new and simple approach for modulating the electromagnetic wave propagating in a dielectric medium in the form of solitons by considering the torque developed between the induced dipoles in the medium and external field without taking into account the nonlinear Kerr effect. A reductive perturbation method deduces the Maxwell equation coupled with the Larmor equation of the torque to the derivative nonlinear Schrödinger equation that admits optical solitons in the medium.

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Hasegawa and Tappert [1] showed that under slowly varying amplitude electromagnetic (EM)-pulse propagating in a nonlinear Kerr fiber medium is governed by the completely integrable nonlinear Schrödinger (NLS) equation $iA_x+A_{tt}+\mu|A|^2A=0$ that admits N-soliton solutions (μ=constant). This was derived from Maxwell equations under the assumption of weak linear dispersion and A is the slowly varying amplitude of the electric field of the electromagnetic wave (EMW). Later this was experimentally verified by Mollenauer *et al* [2]. For more channel handling capacity, it is necessary to transmit pulses at the order of sub-picosecond and femto-second frequency levels. But the propagation of such ultra short pulses (USP) experience higher order effects like third order dispersion (TOD), self steepening (SS) and stimulated Raman scattering (SRS). In this case the wave propagation is described by the higher order nonlinear Schrödinger (HNLS) equation $iA_x+A_{tt}+\mu|A|^2A+i(\mu_1A_{ttt}+\mu_2|A|^2A_t+\mu_3|A|^2_tA)=0$. For example, when $\mu=0$ and when the two inertial contributions of the nonlinear polarization namely, the stimulated Raman scattering (SRS) and self steepening (SS) are equal ($\mu_2=\mu_3=1$) in the absence of third order dispersion (TOD) ($\mu_1=0$), the HNLS reduces to the completely integrable soliton possessing derivative nonlinear Schrödinger (DNLS) equation [3] $iA_x+A_{tt}+i(|A|^2A)_t=0$, and, however, when $\mu\neq 0$, it reduces to the completely integrable mixed derivative nonlinear Schrödinger (MDNLS) equation [4] $iA_x+A_{tt}+\mu|A|^2A+i(|A|^2A)_t=0$ that also admits solitons. When both the TOD and SS effects are included one obtains the Hirota equation for specific parametric choices ($\mu_1=1,\mu_2=\pm6,\mu_3=0$) [5] $iA_x+A_{tt}+\mu|A|^2A+i(A_{ttt}\pm6|A|^2A_t)=0$ which explains the propagation of ultra short pulses in the form of soliton. Later Sasa and Satsuma [6] proved that for a particular choice of parametric relations for TOD, SS and SRS ($\mu_1=1,\mu_2=6,\mu_3=3$), the soliton propagation is supported by another completely integrable HNLS equation $iA_x+A_{tt}+\mu|A|^2A+i(A_{ttt}+6|A|^2A_t+3|A|^2_tA)=0$.

The propagation of optical pulses in birefringent fibers have become very useful in the context of nonlinear directional couplers and a lot of work has been carried out recently, in this direction where the dynamical equations governing the propagation of signals in the form of optical solitons reduce to the two coupled nonlinear Schrödinger (CNLS)

family of equations [7,8] $iA_{jx} + A_{jtt} + \mu[\sum_{k=1}^{2}|A_k|^2]A_j + i[\mu_1 A_{jxxx} + \mu_2(\sum_{k=1}^{2}|A_k|^2)A_{jx} + \mu_3(\sum_{k=1}^{2}(|A_k|^2)_x A_j] = 0$, where $j = 1, 2$. This has been further extended to N-signals [9]. Very recently, the study of propagation in birefringent optical fibers introduced the new concept of shape changing solitons that share energy amongst themselves during propagation [7,8]. This energy switching behaviour of optical solitons has been used for constructing all optical logic gates [10]. The above models (both single and coupled) support propagation of optical pulses in the pico- and femto-second ranges that emerge from high intensity lasers. Now it has also been experimentally proved that visible white light emerging from an incandescent source, namely a quartz-tungsten-halogen bulb when propagating through the photo refractive crystal $Sr_{0.75}Ba_{0.25}Nb_2O_6$, optical solitons are formed by self trapping [11]. In another direction self modulation of quasi-monochromatic EMW into spatially coherent optical solitons in a dielectric medium is described by the NLS equation $iA_t + A_{xx} + 2|A|^2A = 0$ and their higher order and coupled versions, however with an interchange of the time(t) and space(x) variables in their derivatives [12,13].

In this letter we describe a new and simple approach to modulate the EMW propagating in a dielectric medium in the form of spatially coherent optical solitons without taking into account the nonlinear Kerr effect in the medium as has been done so far but only by considering the torque associated between the induced dipole moment of the medium and the electric field component of the EM field in the form of Larmor equation. When the dielectric medium is exposed to the weak EM field the induced dipoles, due to polarization when continuously subjected to the external field, give rise to torque. To the lowest order of perturbation the torque due to the electric field component of the EM field is due to the interaction of it with the induced electric dipole moment of the medium. When the medium is anisotropically polarizable the induced dipole moment $\mathbf{p}$ and the electric field $\mathbf{E}$ are not parallel in general and the torque is given by $\mathbf{\Gamma} = \mathbf{p} \wedge \mathbf{E}$. The angular momentum vector is the only physical quantity that can define a unique direction for the polarised atom of the dielectric medium. The classical time evolution equation for the angular momentum can be constructed by writing the time derivative of the angular momentum as equal to

the above torque. Thus we have the gyroscopic equation or the Larmor equation for the electric field in the form $\frac{\partial \mathbf{P}(\mathbf{r},t)}{\partial t} = \gamma \mathbf{P} \wedge \mathbf{E}$ [14], where the polarization $\mathbf{P}$ is the sum $\sum_i \mathbf{p}_i$ over all the atoms in a unit volume and γ is the Larmor frequency. Even though existence of anisotropic polarizability in the dielectric medium is necessary for the presence of an electric field induced torque, it is not sufficient and therefore in addition, it is required that the angular momentum vector should be neither parallel nor orthogonal to the electric field.

The propagation of EMW in the dielectric, when there are no free stationary and moving electric charges, is described by the Maxwell equations :

$$\nabla \cdot \mathbf{D} = 0, \qquad \nabla \cdot \mathbf{B} = 0, \tag{1a}$$

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \wedge \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \tag{1b}$$

where the fields $\mathbf{E}$, $\mathbf{B}$, $\mathbf{H}$ and $\mathbf{D}$, which are represented by 3-component vectors, bear the usual text book meaning and ϵ is the dielectric constant of the medium. As we consider a dielectric medium for EMW propagation here the constitutive relations between $\mathbf{B}$ and $\mathbf{H}$ can be written as $\mathbf{H} = \frac{\mathbf{B}}{\mu}$, where μ is the magnetic permeability of the medium. Now taking curl on both sides of the first of Eq. (1b) and using the second of Eq.(1b), and after using the relations for $\mathbf{H}$ and $\mathbf{D} = \epsilon\mathbf{E} + \mathbf{P}$, we finally obtain $\frac{\partial^2}{\partial t^2}[\mathbf{E} + \frac{1}{\epsilon}\mathbf{P}] = c^2[\nabla^2\mathbf{E} - \nabla(\nabla \cdot \mathbf{E})]$, where $c = \frac{1}{\sqrt{\mu\epsilon}}$ and $\mathbf{P} = (P^x, P^y, P^z)$. To understand the nature of propagation of EMW in the dielectric medium we now have to solve Maxwell equations coupled with Larmor equation. As fiber is the convenient form of dielectric medium for propagation, it is more meaningful to study the one dimensional version of these equations (say along x-direction). We further consider x as the direction of propagation of EMW, which is taken parallel to the length of the dielectric fiber. Thus we have the following set of coupled equations to be solved.

$$\frac{\partial^2}{\partial t^2}[\mathbf{E}(x,t) + \frac{1}{\epsilon}\mathbf{P}(x,t)] = c^2\left[\frac{\partial^2 \mathbf{E}(x,t)}{\partial x^2} - \frac{\partial^2}{\partial x^2}E^x(x,t)\mathbf{e}\right], \tag{2a}$$

and

$$\frac{\partial \mathbf{P}(x,t)}{\partial t} = \mathbf{P}(x,t) \wedge \mathbf{E}(x,t), \tag{2b}$$

where $\mathbf{e} = (1, 0, 0)$ and time t has been rescaled and velocity c redefined while writing Eq.(2).

Now, we solve the set of coupled equations (2) using a reductive perturbation method originally developed by Taniuti and Yajima [15] to understand the modulation of the slowly varying EMW due to nonlinear interactions in the medium. When the amplitude varies slowly over the period of the oscillation, a stretching transformation allows us to separate the system into a rapidly varying part associated with the oscillation and a slowly varying one such as the amplitude. Nonlinear terms modulate the slowly varying amplitude of the wave. Then a formal solution is given in an asymptotic expansion about a uniform value. A similar study has been performed in the context of EMW propagation in ferromagnetic medium and different types of EM solitons were obtained depending on the nature of the medium [16,17]. We assume that for the set of coupled Eqs.(2) there exists a formal solution expanded asymptotically in terms of a small parameter ε in the neighbourhood of a constant value. For this we expand the components of the electric field and polarization of the medium in terms of a small parameter ε as $\mathcal{F}^x(\xi,\tau) = \mathcal{F}_0 + \varepsilon\mathcal{F}_1^x + \varepsilon^2\mathcal{F}_2^x + \ldots$, $\mathcal{F}^\alpha(\xi,\tau) = \varepsilon^{\frac{1}{2}}\ [\mathcal{F}_1^\alpha + \varepsilon\mathcal{F}_2^\alpha + \ldots]$, where $\alpha = y, z$. Here $\mathcal{F}$ stands for the electric field $\mathbf{E}(x,t)$ as well as for the polarization vector $\mathbf{P}(x,t)$. Due to the anisotropic character of the molecules in the medium as originally assumed here we have expanded the electric field and the polarization vectors in a nonuniform way about the constant values E_0 and P_0 respectively. The expanded electric field and polarization are functions of the slow variables τ and ξ introduced through the stretching of time ($\tau = \varepsilon^2 t$) and the space variable ($\xi = \varepsilon(x - vt)$), to take care of the slow variation in amplitude [18]. Here ε is the same perturbation parameter introduced earlier and v is the group velocity of the propagating EMW.

We now substitute the expansions of $\mathbf{E}$ and $\mathbf{P}$ in the component forms of Eqs.(2) and collect terms proportional to different powers of ε and try to solve the resultant equations. On solving the resultant equations at $O(\varepsilon^0)$, we obtain $E_0 = \frac{-P_0}{\epsilon}$ and $E_1^\alpha = kP_1^\alpha$, where $k \equiv (E_0/P_0) = \frac{c^2}{\epsilon(c^2-v^2)}$. Then solving them at $O(\varepsilon^1)$, after using the results of $O(\varepsilon^0)$ we finally obtain $E_1^x = \frac{-P_1^x}{\epsilon}$ and

$$\frac{\partial E_1^x}{\partial \xi} = \frac{\hat{\gamma}}{\epsilon v k}\left[E_1^y \int_{-\infty}^{\xi} \frac{\partial E_1^z}{\partial \tau} d\xi' - E_1^z \int_{-\infty}^{\xi} \frac{\partial E_1^y}{\partial \tau} d\xi'\right], \tag{3a}$$

$$\frac{\partial E_1^y}{\partial \xi} = \frac{\hat{\gamma} k P_0}{v}\int_{-\infty}^{\xi} \frac{\partial E_1^z}{\partial \tau} d\xi' - \frac{(1+\epsilon k)}{v} E_1^z E_1^x, \tag{3b}$$

$$\frac{\partial E_1^z}{\partial \xi} = \frac{(1+\epsilon k)}{v} E_1^y E_1^x - \frac{\hat{\gamma} k P_0}{v}\int_{-\infty}^{\xi} \frac{\partial E_1^y}{\partial \tau} d\xi', \tag{3c}$$

where $\hat{\gamma} = \frac{2v(2c^2-v^2)}{c^2(v^2-c^2)}$. Now, we define

$$\psi = (E_1^y - iE_1^z), \qquad \mid \psi \mid^2 = E_1^x. \tag{4}$$

The above definitions suggest that $E_1^x = E_1^{y2} + E_1^{z2}$, which is in accordance with the nonuniform expansion. Eqs.(3b) and (3c), after a single differentiation, can be combined together to give after some algebra and rescaling the variable τ the following single equation in terms of the newly defined variable ψ,

$$i\psi_\tau + \psi_{\xi\xi} + i\lambda[\mid \psi \mid^2 \psi]_\xi = 0. \tag{5}$$

where $\lambda = \frac{-(1+\epsilon k)kP_0}{v}$. It may be verified that on using the definitions (4), Eq.(3a) can be written in the form of Eq.(5). Eq.(5) is the completely integrable derivative nonlinear Schrödinger (DNLS) equation which admits N-soliton solutions. The multi-soliton solutions of Eq.(5) can be obtained in the framework of the inverse scattering transform (IST) method [3]. It is moreover easy and straightforward to construct the N-solitons using Hirota's bilinearisation procedure [19]. Following ref. [20], we consider the nonlinear transformation $\psi = \frac{gf^*}{f^2}$, so that Eq.(5) can be written in terms of the following bilinear equations.

$$(iD_\tau + D_\xi^2) g.f = 0, \tag{6a}$$

$$\begin{aligned} &\left[(g.f)(iD_\tau + D_\xi^2) + 2D_\xi(g.f)D_\xi\right](f^*.f) \\ &-2(g.f^*)D_\xi^2(f.f) + i\lambda\left[3D_\xi(g.f) + gfD_\xi\right](g^*.g) = 0, \end{aligned} \tag{6b}$$

where $D_\tau^m \ D_\xi^n \ (g \cdot f) = \left[\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \tau'}\right]^m \ \left[\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \xi'}\right]^n \ g(\xi,\tau) \cdot f(\xi',\tau')|_{\text{at } \xi'=\xi,\ \tau'=\tau}$ are Hirota's bilinear operators [19]. In order to obtain soliton solutions, we assume the series expansions for g, f and f^* as $g = \chi g_1 + \chi^3 g_3 + ...$, $f = 1 + \chi^2 f_2 + \chi^4 f_4 + ...$ and $f^* = 1 + \chi^2 f_2^* + \chi^4 f_4^* + ...$ where χ is an arbitrary parameter. Now, for example to construct a one-soliton solution we

set $g = \chi g_1$, $f = 1 + \chi^2 f_2$ and $f^* = 1 + \chi^2 f_2^*$ and then collect terms with similar powers of χ and solve the resultant equations which admit the following solutions for g, f and f^*.

$$g(\xi, \tau) = exp[\eta_1]. \tag{7a}$$

$$f(\xi, \tau) = 1 + \frac{i\lambda\Omega_1}{2(\Omega_1 + \Omega_1^*)^2} exp[\eta_1 + \eta_1^*], \tag{7b}$$

$$f^*(\xi, \tau) = 1 - \frac{i\lambda\Omega_1^*}{2(\Omega_1 + \Omega_1^*)^2} exp[\eta_1 + \eta_1^*], \tag{7c}$$

where $\eta_1 \equiv \eta_{1R} + i\eta_{1I} = K_1\tau + \Omega_1\xi + \eta_1^{(0)}$, $K_1 = i\Omega_1^2$ and $\Omega_1 = \Omega_{1R} + i\Omega_{1I}$, and the constant $\eta_1^{(0)} = \eta_{1R}^{(0)} + i\eta_{1I}^{(0)}$. Thus the one soliton solution for $\psi = \frac{gf}{f^*}$ is found to be

$$\psi = Qsech(\eta_{1R} + A_0)tanh(\eta_{1R} + A_0), \tag{8}$$

where $Q = \frac{-1}{2} exp[i\eta_{1I} + A]$, $A_0 = \eta_{1R}^{(0)} + A$ and $A = \frac{1}{2} ln(\frac{i\lambda\Omega_1}{8\Omega_{1R}^2})$. Similarly we can find two, three ... and N-soliton solutions. As the details of constructing multi-soliton solutions are very lengthy and the form of N-soliton solution is very cumbersome we are not presenting the details here and interested readers can refer to ref. [20]. Using the one soliton solution given in Eq.(8) in the definition of ψ (Eqs.(4)) the components of the electric field vector can be constructed. Thus the optical one soliton in terms of the electric field vector component of the EM field can be written as

$$E_1^x = \frac{-P_1^x}{\epsilon} = \frac{1}{4} exp(2A) sech^2(\eta_{1R} + A_0) tanh^2(\eta_{1R} + A_0), \tag{9a}$$

$$E_1^\alpha = kP_1^\alpha = -\frac{1}{4} exp(A)[Re(Q)\delta_{x\alpha} - iIm(Q)\delta_{y\alpha}] sech(\eta_{1R} + A_0) tanh(\eta_{1R} + A_0), \tag{9b}$$

where $Re(Q)$ and $Im(Q)$ represent the real and imaginary parts of Q respectively and $\delta_{x\alpha}$ and $\delta_{y\alpha}$ are Kronecker delta functions.

Eqs.(9) show that the electric field component of the propagating plane EMW is modulated in the form of solitons by compensating the dispersion with the torque produced by the rotation of the dipoles induced in the medium. In both Hasegawa's theoretical model

FIGURES

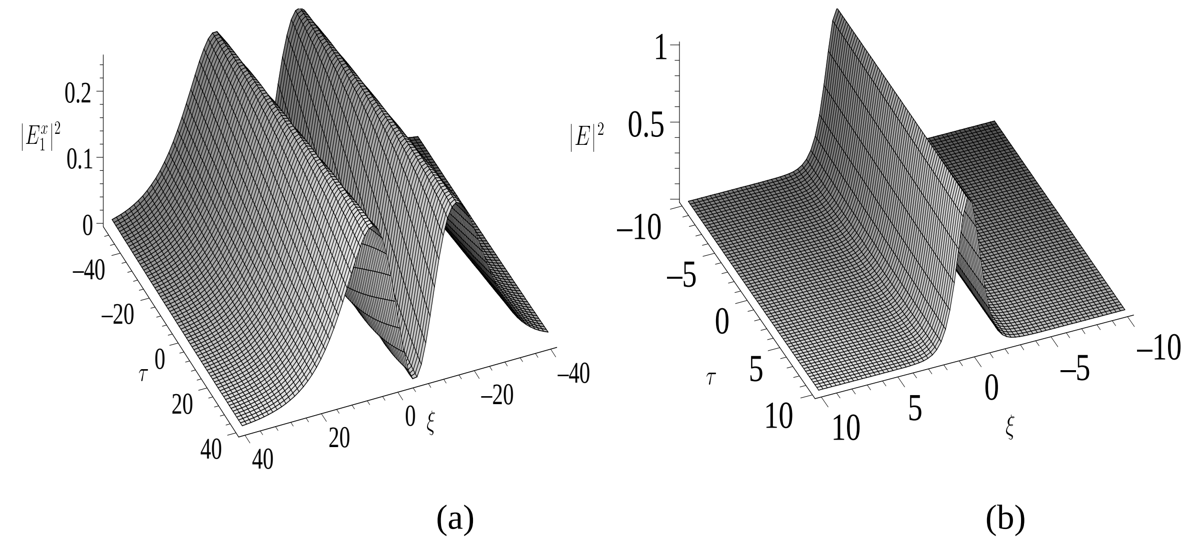

FIG. 1. (a) x-component of $E_1(E_1^x)$ exhibiting soliton behaviour with two maxima (twin pulses) when $\Omega_{1R} = 0.08$, $\Omega_{1I} = 0.003$, $\eta_{1R} = 0.02$, $\lambda = 0.1$ and $A = 0.3010$, (b) the form of NLS one-soliton for comparison.

for optical soliton in the pico-second region (temporal coherence) and in the case of Zhakarov/Manakov's model for modulation of the EMW in the form of solitons (spatial coherence), the soliton is governed by NLS equation.

However, in our problem where the Kerr nonlinearity is not taken into account, but, by getting support for nonlinearity from the rotation of induced dipoles in the medium due to the torque with the external field the soliton is governed by the DNLS equation. The spatial soliton is formed when the EMW induces a waveguide in the dielectric medium via the nonlinearity of the torque induced, self trapped and in turn is guided in its own wave guide. Unlike the NLS-one soliton, here we have multihump (two) soliton (see Figs.(1a) and (1b)). In nonlinear optics, multihump solitons corresponding to multimodes propagating without interference and loosing their stability are known [21,22]. Even though we get multihump (two humps) solitons, in our case it corresponds to a single mode which is again a stable soliton corresponding to the completely integrable DNLS equation.

Now to understand the effect of the nonlinear term $(|\psi|^2\psi)_\xi$ in the optical soliton formation during propagation of EMW we consider Eq.(5) by dropping the dispersion term [23].

$$\psi_\tau + \lambda(|\psi|^2\psi)_\xi = 0. \tag{10}$$

Assuming the solution in the form of $\psi(\xi,\tau) = q(\xi,\tau)exp[i\theta(\xi,\tau)]$, Eq.(10) can be written after rescaling τ suitably as $q_\tau \; + 3q^2 q_\xi \; = \; 0$, $\theta_\tau \; + q^2\theta_\xi \; = \; 0$, for which the solution can be written in the implicit form. Infact the first of the above equations is equivalent to the dispersionless modified Korteweg - de Vries (mKdV) equation for which the solution can be written in the implicit form as $q^2 = a(\xi - 3\tau q^2)$ or equivalently in another form as $\xi = 3\tau q^2 + b(q^2)$, where a is an arbitrary function determined by the initial profile and b is the inverse function $b = \frac{1}{a}$. The second equation of the above equations can be solved using the method of characteristics and again the solution can be written in terms of another arbitrary function. Explicit solutions can be found for specific initial pulse profiles. For example, if we choose a Gaussian pulse in the form $q^2(\xi,0) = exp(\frac{-\xi^2}{\xi_0^2})$ and using this, we find that $a(\xi) = exp(\frac{-\xi^2}{\xi_0^2})$ and hence $b(q^2) = \pm\xi_0(ln\frac{1}{\rho^2})^{\frac{1}{2}}$. This leads to the solution $q^2 = exp[-\frac{(\xi-3\tau q^2)^2}{\xi^2}]$ which gives $\xi = 3\tau q^2 \pm \xi_0(ln\frac{1}{\rho^2})^{\frac{1}{2}}$, where the signs $(+)$ and $(-)$ refer to the trailing and leading edges of the soliton respectively. From the above expression for ξ it is clear that the local increment in velocity is inversely proportional to the square of the amplitude thus leading to a shift of the pulse peak to the trailing edge. As in the case of mKdV equation here also the self-steepening of the trailing edge leads to the appearance of an EM shock wave. If we repeat the calculations for a sech-profile in the form $q^2(\xi,0) = sech^2(\frac{\xi}{\xi_0})$, the results show the formation of a shock EMW at the trailing edge of the pulse as in the Gaussian case. Further, if we choose a periodic wave profile in the form $q^2(\xi,0) = cos(\frac{\xi}{\xi_0})$, once again we obtain the same result exhibiting shocks in the negative slope region of the propagating EMW. This EM shock is compensated by the dispersion to form a localized spatially coherent optical soliton.

In this paper we have proposed a simple and new approach to generate optical solitons in

a dielectric medium during propagation of EMW without taking into account the nonlinear Kerr effect as done normally. On the other hand we considered the torque produced in the medium due to the rotation of the induced dipoles by the interacting exterior electric field component of the EM field and constructed the Larmor equation for precession of the induced dipoles (or electric polarization) in the medium. This is then solved with Maxwell equations which govern the propagation of EMW in the dielectric medium using a reductive perturbation method. The equations were reduced to the integrable derivative nonlinear Schrödinger equation that admits N-soliton solutions. Here the nonlinearity required to support optical soliton is provided by the torque developed between the induced electric dipoles in the medium and the EM field and not from the Kerr effect. Another important difference, when comparing the results of the usual optical soliton study, is that, unlike the other case here we get multihumped optical solitons of the DNLS equation. A careful analysis of the final results showed that the nonlinearity due to the torque actually produced a shock in the propagating EMW and when it is balanced by the linear dispersion, the plane EMW is modulated into spatially coherent optical solitons. Also, the polarization of the medium is excited in the form of solitons. We expect our method to bring out fascinating results of energy sharing optical solitons and optical logic gate operations in birefringent fiber medium in a straightforward way through this analytic approach. Further, this is expected to be useful in analysing all higher order effects in polarization of the medium.

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